Dyer-Lashof Operations

Guozhen Wang

Shanghai Center for Mathematical Sciences

March 30, 2018

Contents

Actions and co-actions

Let A be a generalised cohomomoly theory, represented by a spectrum which we also denote by A.

By Yoneda's lemma, the set of self natural transformations on A is the self-maps $[A, A]_*$ on A which is also the A-cohomology $A^*(A)$ of the spectrum A.

For example, when $A = H$ is the mod p ordinary cohomology, we get the Steenrod algebra H^*H.

Dually, we have the dual co-action of A_*A on the A-homology, provided A is a ring spectrum and A_*A is flat over A_*: for any spectrum X, we have the co-action

$$A_*X = \pi_*(A \wedge X) \xrightarrow{id \wedge u \wedge id} \pi_*(A \wedge A \wedge X) \xleftarrow[\cong]{id \wedge m \wedge id} A_*A \otimes_{A_*} A_*(X)$$

Structure of the ring of co-operations

Let A be an E_∞-ring spectrum, with unit $u : S \to A$ and product $m : A \wedge A \to A$. Then we can construct the following maps involving $\pi_* A = A_*$ and $\pi_*(A \wedge A) = A_* A$:

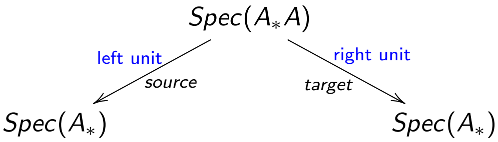
- product: $\pi_*(A \wedge A) \otimes \pi_*(A \wedge A) \to \pi_*(A \wedge A \wedge A \wedge A) \xrightarrow{id \wedge shuffle \wedge id} \pi_*(A \wedge A \wedge A \wedge A) \xrightarrow{m \wedge m} \pi_*(A \wedge A)$.
- left unit: $\pi_* A \xrightarrow{id \wedge u} \pi_*(A \wedge A)$.
- right unit: $\pi_* A \xrightarrow{u \wedge id} \pi_*(A \wedge A)$.
- co-product: $\pi_*(A \wedge A) \xrightarrow{id \wedge u \wedge id} \pi_*(A \wedge A \wedge A)$.
- co-unit: $\pi_*(A \wedge A) \xrightarrow{m} \pi_* A$.
- conjugation: $\pi_*(A \wedge A) \xrightarrow{shuffle} \pi_*(A \wedge A)$.

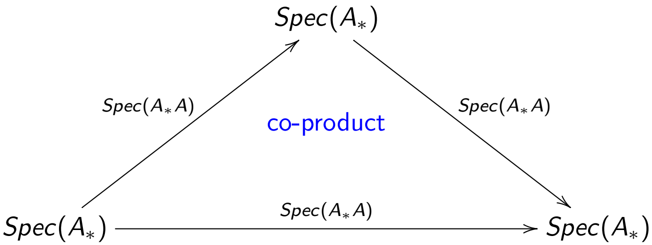
If $\pi_*(A \wedge A)$ is flat over $\pi_* A$, then the following map is an isomorphism:

$$\pi_*(A \wedge A) \otimes_{\pi_* A} \pi_*(A \wedge A) \to \pi_*(A \wedge A \wedge A \wedge A) \xrightarrow{id \wedge m \wedge id} \pi_*(A \wedge A \wedge A)$$

Hopf algebroids

We assume $\pi_*(A \wedge A)$ is flat over $\pi_* A$ from now on. Then (A_*, A_*A) forms a Hopf algebroid, i.e. a groupoid object in the category of affine schemes.





The pairing of the Steenrod algebra with its dual

We can determine the formula for the pairing between the Steenrod algebra, which is the cohomology of H, and its dual, the homology of H.

Consider the map

$$s : (\Sigma^{\infty-1}\mathbb{R}P^{\infty})^{\wedge n} \to H^{\wedge n} \to H$$

Which sends the class $e_{2^{i_1}} \otimes \cdots \otimes e_{2^{i_k}}$ to $\xi_{i_1} \ldots \xi_{i_k}$.

Recall that $< s^*(x), y >=< x, s_*(y) >$ for any $x \in H_*((\Sigma^{\infty-1}\mathbb{R}P^{\infty})^{\wedge n})$ and $y \in H^*(H)$.

So we get

$$< Sq^i((e_1^*)^{\otimes k}), e_{2^{i_1}} \otimes \cdots \otimes e_{2^{i_k}}) >=< Sq^i, \xi_{i_1} \ldots \xi_{i_k} >$$

and the Cartan formula implies that

$$< Sq^i, \xi_1^{r_1} \ldots \xi_k^{r_k} >= \begin{cases} 1, & \text{if } \xi_1^{r_1} \ldots \xi_k^{r_k} = \xi_1^i \\ 0, & \text{otherwise} \end{cases}$$

Action of the Steenrod algebra

The dual action of the Steenrod algebra can be determined by the co-product. Recall that the action of the Steenrod algebra are determined by the formula

$$Sq^i_* x =< Sq^i \otimes 1, \psi(x) >$$

$$Sq^r_* \zeta_i = \begin{cases} \zeta_{i-k}^{2^k}, & \text{if } r = 2^k - 1 \\ 0, & \text{otherwise} \end{cases}$$

Odd primary case

For an odd prime p, we have

$$H_*H \cong \mathbb{F}_p[\xi_1, \xi_2, \dots] \otimes E[\tau_0, \tau_1, \dots]$$

$$\psi(\xi_i) = \sum \xi_{i-k}^{p^k} \otimes \xi_k$$

$$\psi(\tau_i) = \tau_i \otimes 1 + \sum \xi_{i-k}^{p^k} \otimes \tau_k$$

Power operations

Let A and E be E_∞-ring spectra. Then $A \wedge E$ is also E_∞ and in particular we have the structure maps

$$\mu_n : D_n(A \wedge E) \to A \wedge E$$

where $D_n(X) = X^{\wedge n} \wedge_{\Sigma_n} E\Sigma_{n+}$ is the extended power.
Let $x : S^m \to A \wedge E$ be an element of $A_m E$. We can construct the following map:

$$\tilde{x} : A \wedge D_n S^m \xrightarrow{id \wedge D_n x} A \wedge D_n(A \wedge E) \xrightarrow{id \wedge \mu_n} A \wedge A \wedge E \xrightarrow{m \wedge id} A \wedge E$$

For any $e \in A_k(D_n S^m)$, we can define the power operation $\Theta^e : A_m E \to A_k E$ by

$$\Theta^e(x) = \tilde{x}_*(e)$$

The Dyer-Lashof operations

We consider the case when $A = H$ is the Eilenberg-MacLane spectrum.

Recall that D_2S^m is homotopy equivalent to $\Sigma^m\mathbb{R}P_m^\infty$.

Let $e_r \in H_{r+2m}(D_2S^m)$ be the generator in degree $r + 2m$.

Define the Dyer-Lashof operations $Q^r, r \in \mathbb{Z}$ to be

$$Q^r(x) = \Theta^{e_{r-m}}x$$

for x in degree m.

From the definition, we have

- $Q^r(x) = 0$ for $r < m$.
- $Q^m(x) = x^2$.

Cartan formula

$$Q^r(xy) = \sum_{i+j=r} (Q^i x)(Q^j y)$$

Adem relations

If $r > 2s$ then

$$Q^r Q^s = \sum_i \binom{i-s-1}{2i-r} Q^{r+s-i} Q^i$$

Nishida relations

$$Sq_*^r Q^s = \sum_i \binom{2^n+s-r}{r-2i} Q^{s-r+i} Sq_*^i$$

for any large enough n.

Dyer-Lashof operations on the dual Steenrod algebra

Let $\xi = t + \xi_1 t^2 + \xi_2 t^4 + \cdots + \xi_k t^{2^k} + \cdots$.

(Steinberg)

- $t^{-1} + \xi_1 + \sum_{s>0} Q^s \xi_1 = \xi^{-1}$.
- $Q^{2^i-2}\xi_1 = \zeta_i$.
- For $s > 0$, we have

$$Q^s \zeta_i = \begin{cases} Q^{s+2^i-2}\xi_1, & \text{if } s \equiv 0 \ or \ -1 \mod 2^i \\ 0, & \text{otherwise} \end{cases}$$

- $Q^{2^i}\zeta_i = \zeta_{i+1}$.

The action of the Dyer-Lashof algebra on the dual Steenrod algebra can be shown using the Nishida relations.

We recall the following proposition:

The following are equivalent for $x, y \in A_*$ with $|x| = |y| > 0$:

- $x = y$.
- For any $P \in A^*$, $< P, x > = < P, y >$.
- For any admissible $I = (i_1, i_2, \dots)$ with $|I| = |x|$, $Sq^I_* x = Sq^I_* y$.
- For any $P \in A^*$, $P_* x = P_* y$.
- For any $k \geq 0$, $Sq^{2^k}_* x = Sq^{2^k}_* y$.

Computation of $Q^{2^i}\zeta_i$

By the Nishida relation, we have

$$Sq_*^1 Q^{2^i}\zeta_i = Q^{2^i-1}\zeta_i = \zeta_i^2$$

and for $k > 0$,

$$Sq_*^{2^k} Q^{2^i}\zeta_i = \binom{2^k+2^i}{2^i} Q^{2^i-2^k}\zeta_i = 0$$

On the other hand,

$$Sq_*^{2^k}\zeta_{i+1} = \begin{cases} \zeta_i^2, & k=0 \\ 0, & \text{otherwise} \end{cases}$$

Hence

$$Q^{2^i}\zeta_i = \zeta_{i+1}$$

Proof of Nishida relations

The Nishida relation follows from the following computation of Steenrod operations on extended powers:

Let X be a spectrum. Denote $e_n \in H_n(B\Sigma_2)$ be the generator. Then

1. $H_*(X^{\wedge 2} \wedge_{\Sigma_2} E\Sigma_{2+}) \cong H_*(\Sigma_2, H^*(X))$.
2. If $x \in H_q(X)$, then $Sq_*^r(e_s \otimes x^2) = \sum_i \binom{s+q-r}{r-2i} e_{s-r+2i} \otimes Sq_*^i(x)^2$.

To prove the Nishida relation, recall that $Q^s(x) = \mu_*(e_{s+q} \otimes x^2)$.
Hence $Sq_*^r Q^s(x) = \sum_i \binom{s-r}{r-2i} Q^{s-r+i} Sq_*^i(x)$.

Steenrod operations on extended powers

Let X be a suspension spectrum.

We can consider the diagonal map

$$d : X \wedge B\Sigma_{2+} \to X^{\wedge 2} \wedge_{\Sigma_2} E\Sigma_{2+}$$

From the definition of Steenrod powers, we can show that

$$d_*(e_r \otimes x) = \sum_k e_{r+2k-s} \otimes (Sq_*^k x)^2$$

So we get

$$e_{r-s} \otimes x^2 = d_*(e_r \otimes x) + \sum_{k>0} e_{r+2k-s} \otimes (Sq_*^k x)^2$$

The action of the Steenrod algebra on $H_*(X^{\wedge 2} \wedge_{\Sigma_2} E\Sigma_{2+})$ can be computed inductively using the Cartan formula and Adem relations.

When X is a finite CW spectrum, we use the fact

$$X^{\wedge 2} \wedge_{\Sigma_2} E\Sigma_{2+} \cong \Sigma^{-1}(\Sigma X)^{\wedge 2} \wedge_{\Sigma_2} Th(-\gamma)$$

where $Th(-\gamma)$ is the Thom spectrum of the negative tautological bundle.
Then we can reduce to the case of suspension spectrum by considering $\Sigma^k X$ for some k and use James periodicity:

$Th(\gamma^n|_{S^m})$ is Σ_2-equivalently homotopy equivalent to $\Sigma^{-2^k} Th(\gamma^{n+2^k}|_{S^m})$ for some large enough k depending on m.

Another formulation of the co-action

We introduced the co-action as the map:

$$\psi : \pi_*(A \wedge A) \xrightarrow{id \wedge u \wedge id} \pi_*(A \wedge A \wedge A) \xleftarrow[\cong]{\pi_*(id \wedge id \wedge u) \otimes \pi_*(u \wedge id \wedge id)} \pi_*(A \wedge A) \otimes_{\pi_* A} \pi_*(A \wedge A)$$

where in the rightmost, the left $\pi_*(A \wedge A)$ is viewed as a $\pi_* A$-module via the right unit and the right $\pi_*(A \wedge A)$ is viewed as a $\pi_* A$-module via the left unit.

There is another formulation of the co-action:

$$\tilde{\psi} : \pi_*(A \wedge A) \xrightarrow{u \wedge shuffle} \pi_*(A \wedge A \wedge A) \xleftarrow[\cong]{\pi_*(id \wedge id \wedge u) \otimes \pi_*(id \wedge u \wedge id)} \pi_*(A \wedge A) \otimes_{\pi_* A} \pi_*(A \wedge A)$$

where in the rightmost, both $\pi_*(A \wedge A)$ are viewed as a $\pi_* A$-module via the left unit. This is the usual identification using the universal coeficent formula.

The two co-actions are related as

$$\tilde{\psi} = switch \circ (\chi \otimes id) \circ \psi$$

Power operations and co-actions

Let B be another E_∞-ring spectrum. Then power operations for A and B are related the the following way:

$$
\begin{array}{ccccccc}
A \wedge D_n S^m & \xrightarrow{id \wedge D_n x} & A \wedge D_n(A \wedge E) & & \xrightarrow{id \wedge \mu_n} & & A \wedge A \wedge E \\
\Big\downarrow{\scriptstyle id \wedge u \wedge id} & & \Big\downarrow{\scriptstyle id \wedge u \wedge D_n(u \wedge id)} & & & & \Big\downarrow{\scriptstyle id \wedge u \wedge id} \\
A \wedge B \wedge D_n S^m & \xrightarrow{id \wedge D_n(u \wedge x)} & A \wedge B \wedge D_n(B \wedge A \wedge E) & \xrightarrow{id \wedge \mu_n} & A \wedge B \wedge B \wedge A \wedge E & \xrightarrow{id \wedge m \wedge id} & A \wedge B \wedge A \wedge E
\end{array}
$$

In particular, when $A = B$, we have the following formula:

$$\tilde{\psi}(\Theta^e(x)) = \sum_i (1 \otimes \chi(\theta_i))\Theta^{e_i}(\tilde{\psi}(x))$$

where $\psi(e) = \sum_i \theta_i \otimes e_i$.

The coaction formula for the Steenrod squares

We can apply the general coaction formula for the case $A = H$, which is the dual version of the Nishida relations.

Coaction on the homology of $\mathbb{R}P^\infty$

Let $\xi = t + \xi_1 t^2 + \xi_2 t^4 + \cdots + \xi_k t^{2^k} + \cdots$.

For $e_n \in H_n(\mathbb{R}P^\infty)$ the generator, we have

$$\sum_n t^n \psi(e_n) = \sum_s \xi^s \otimes e_s$$

Hence we have:

Let $\zeta = t + \zeta_1 t^2 + \zeta_2 t^4 + \cdots + \zeta_k t^{2^k} + \cdots$.

$$\sum_r \tilde{\psi}(Q^r x) t^r = \sum_k Q^k(\tilde{\psi} x)(1 \otimes \zeta^k)$$

Odd primary Dyer-Lashof algebra

For an odd prime p, we have the Dyer-Lashof operators Q^i and βQ^i.

Adem relations

If $r > ps$,

$$Q^r Q^s = \sum_i (-1_{)}^{r+i} \binom{(p-1)(i-s)-1}{pi-r} Q^{r+s-i} Q^i$$

If $r \geq ps$,

$$\begin{aligned} Q^r \beta Q^s = & \sum_i (-1)^{r+i} \binom{(p-1)(i-s)}{pi-r} \beta Q^{r+s-i} Q^i \\ & + \sum_i (-1)^{r+i} \binom{(p-1)(i-s)-1}{pi-r-1} Q^{r+s-i} \beta Q^i \end{aligned}$$

Odd primary Nishida relation

$$
\begin{aligned}
P_*^r Q^s = & \sum_i (-1)^{r+i} \binom{p^n + (p-1)(s-r)}{r-pi} Q^{s-r+i} P_*^i \\
P_*^r \beta Q^s = & \sum_i \binom{p^n + (p-1)(s-r) - 1}{r-pi} \beta Q^{s-r+i} P_*^i \\
& - \sum_i \binom{p^n + (p-1)(s-r) - 1}{r-pi-1} Q^{s-r+i} P_*^i \beta
\end{aligned}
$$

for any large enough n.

References

- A. Baker, Power operations and coactions in highly commutative homology theories
- J. P. May, A general algebraic approach to Steenrod operations
- J. Milnor, The Steenrod algebra and its dual
- G. Nishida, Cohomology operations in iterated loop spaces
- M. Steinberger, Homology operations for H_∞ and H_n spectra