Prisms and Topological Cyclic Homology of Integers

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Topological Hochschild Homology

Let A be an E_∞-ring spectrum.

$$THH(A) = A^{\wedge S^1}$$

is the free S^1-E_∞-ring spectrum generated by A.
There is a cyclotomic structure defined on $THH(A)$, i.e. an E_∞-homomorphism

$$\varphi : THH(A) \to THH(A)^{tC_p}$$

equivariant with respect to the group isomorphism $S^1 \cong S^1/C_p$.

Topological Cyclic Homology

Topological periodic homology

$$TP(A) = (THH(A)^{tC_p})^{hS^1}$$

Topological negetive cyclic homology

$$TC^-(A) = THH(A)^{hS^1}$$

Topological cyclic homology

$TC(A)$ is the equalizer of the canonical map

$$can : TC^-(A) \to TP(A)$$

and the Frobenius

$$\varphi : TC^-(A) \to TP(A)$$

Cyclotomic Trace

The cyclotomic trace map gives an homomorphism of E_∞-ring spectra:

$$tr : K(A) \to TC(A)$$

(Dundas-Goodwillie-McCarthy)

Let $A_0 \to A_1$ be a homomorphism of connective E_∞-ring spectra with nilpotent kernel on π_0. Then the induced map on the fibers of the cyclotomic trace maps

$$tr : K(A_i) \to TC(A_i),\ i = 0, 1$$

is a stable homotopy equivalence.

Quasi-syntomic Descent Method

We will introduce the quasi-syntomic desecnt method to study topological cyclic homology.

- TP, TC^-, TC are sheaves on the quasi-syntomic site.
- Quasi-syntomic rings (e.g. p-complete local complete intersection Noetherian rings) can be covered by quasiregular semiperfectoid rings.
- The Tate and homotopy fixed point spectral sequences for quasiregular semiperfectoid rings collaps at the E_2-term.

We will further understand:

1. The extension problems for topological periodic homology of quasiregular semiperfectoid rings.
2. Cohomology of the sheaf $TP_*(-)$ in the quasi-syntomic site.
3. The action of the Frobenius.

Perfectoids

A ring R is perfectoid if

1. R is p-adically complete.
2. There is some $\pi \in R$ such that

   $$\pi^p = \rho p$$

   for some unit $\rho \in R^\times$.
3. The Frobenius map φ on R/p is surjective.
4. The kernel of Fontaine's map θ is generated by one element.

Quasiregular Semiperfectoid Rings

A ring S is quasiregular semiperfectoid if

1. S is p-complete with bounded p^∞-torsion.
2. The derived cotagent complex $L_{S/\mathbb{Z}_p}$ has p-complete *Tor* amplitude in $[-1, 0]$.
3. There exists a map $R \to S$ with R perfectoid.
4. The Frobenius of S/p is surjective.

Prisms

Let S be a quasiregular semiperfectoid ring.

Prisms of S

$$\hat{\Delta}_S = TP_0(S)$$

The Frobenius

$$\phi : \hat{\Delta}_S \xrightarrow[\cong]{can^{-1}} TC_0^-(S) \xrightarrow{\varphi} \hat{\Delta}_S$$

Fontaine's Period Ring

Let R be a perfectoid.

$$R^\flat = \varprojlim_{\phi} R/p$$

$$A_{inf}(R) = W(R^\flat)$$

Fontaine's map

$$\theta : A_{inf}(R) \to R$$

sends $[(x_0, x_1, \ldots,)] \in A_{inf}(R)$ to $\lim_{i\to\infty} x_i^{p^i}$.

(Bhatt-Morrow-Scholze)

$$\hat{\Delta}_R \cong A_{inf}(R)$$

and the Frobenius map on $\hat{\Delta}_R$ corresponds to the Frobenius on $A_{inf}(R)$.

TC of Perfectoids

(Bhatt-Morrow-Scholze)

Let $\xi \in \hat{\Delta}_R$ be a generator of the kernel of θ. One can choose elements $u \in TC_2^-(R)$, $\nu \in TC_{-2}^-(R)$, $\sigma \in TP_2(R)$, such that

$$TC^-(R) = A_{inf}[u, \nu]/(u\nu - \xi)$$

$$TP(R) = A_{inf}[\sigma, \sigma^{-1}]$$

with Frobenius

$$\varphi(u) = \sigma$$

$$\varphi(\nu) = \phi(\xi)\sigma^{-1}$$

and canonical map

$$can(u) = \xi\sigma$$

$$can(\nu) = \sigma^{-1}$$

TC of Quasiregular Semiperfectoid rings

Let S be a quasiregular semiperfectoid ring.

- $TC_*^-(S)$ and $TP_*(S)$ are concentrated in even degrees.
- The canonical map

$$can : TC_*^-(S) \to TP_*(S)$$

is injective, and is an isomorphism on degrees ≤ 0.

Nygaard filtration

- An element in $TP_{2k}(S)$ is said to have Nyggard filtration i, if it is detected by an element in $E_\infty^{2i,2i-2k}$ of the Tate spectral sequence.
- $$TC_{2i}^-(S) \cong \mathcal{N}^{\geq i} TP_{2i}(S)$$

Prisms of Quasiregular Semiperfectoid rings

The graded pieces of the Nygaard filtration on $\hat{\Delta}_S$ are isomorphic to topological Hochschild homology:

$$\mathcal{N}^i(\hat{\Delta}_S) \cong THH_{2i}(S)$$

Moreover, we have:

(?)

$\hat{\Delta}_S$ has the structure of a δ-ring.

Recall that $\delta(x) = \frac{x^p - \phi(x)}{p}$, and δ-rings are commutative rings equipped with the operation δ.

Resolution of $\mathbb{Z}_p$

Suppose p is an odd prime. Let $R = \mathbb{Z}_p[p^{\frac{1}{p^\infty}}]$. We have the resolution:

$$\mathbb{Z}_p \to R \to R^{\otimes 2} \to R^{\otimes 3} \to \dots$$

- R is a perfectoid.
- $R^{\otimes i}$ are quasiregular semiperfectoid rings.

(Bhatt-Morrow-Scholze)

THH, TP, TC^-, TC have flat desecnt.

$$TP(\mathbb{Z}_p) \cong \lim_{\Delta} TP(R^{\otimes \bullet})$$

$$TC^-(\mathbb{Z}_p) \cong \lim_{\Delta} TC^-(R^{\otimes \bullet})$$

The BMS Spectral Sequence

There are spectral sequences

$$E_1^{i,j} = TP_i(R^{\otimes j}) \Rightarrow TP_{i-j}(\mathbb{Z}_p)$$

$$E_1^{i,j} = TC_i^-(R^{\otimes j}) \Rightarrow TC_{i-j}^-(\mathbb{Z}_p)$$

Let $S = R^{\otimes 2}$.

1. $(\hat{\triangle}_R, \hat{\triangle}_S)$ forms a Hopf algebroid.
2. $TP_*(R)$ is a $\hat{\triangle}_S$-comodule.
3. The complex $TP_*(R^{\otimes \bullet})$ is isomorphic to the cobar complex

$$C^\bullet(TP_*(R), \hat{\triangle}_S, \hat{\triangle}_R)$$

4. We have the spectral sequence:

$$Ext_{\hat{\triangle}_S}(\hat{\triangle}_R, TP_*(R)) \Rightarrow TP_*(\mathbb{Z}_p)$$

Hopf algebroid structure

units

$$\eta_L : \hat{\Delta}_R \xrightarrow{id \otimes u} \hat{\Delta}_S$$

$$\eta_R : \hat{\Delta}_R \xrightarrow{u \otimes id} \hat{\Delta}_S$$

co-multiplication

$$\psi : \hat{\Delta}_S \xrightarrow{id \otimes u \otimes id} \hat{\Delta}_{R^{\otimes 3}} \xleftarrow{\cong} \hat{\Delta}_S \otimes_{\hat{\Delta}_R} \hat{\Delta}_S$$

where the last isomorphism come from

$$THH(R^{\otimes 3}) \cong THH(S) \wedge_{THH(R)} THH(S)$$

and flatness of $THH_*(S)$.

Structure of $\hat{\Delta}_R$

- $$\hat{\Delta}_R = \mathbb{Z}_p[[z^{\frac{1}{p^\infty}}]]$$
- $$\theta(z) = p$$
- $$\varphi(z) = z^p$$
- $$\xi = z - p$$

Structure of $\hat{\Delta}_S$

Let

$$x = \eta_L(z)$$

$$y = \eta_R(z)$$

$\hat{\Delta}_S$ is multiplicatively generated over $\mathbb{Z}_p[[x^{\frac{1}{p^\infty}}, y^{\frac{1}{p^\infty}}]]$ by the elements:

$$\frac{(x-y)^p}{p} + \text{higher terms}$$

$$\frac{(x-y)^{p^2}}{p^{p+1}} + \text{higher terms}$$

$$\frac{(x-y)^{p^3}}{p^{p^2+p+1}} + \text{higher terms}$$

$$\cdots\cdots$$

Structure of $\hat{\Delta}_S$

There exists a unit $e \in \hat{\Delta}_S$ such that

$$y^p - p = e(x^p - p) \tag{1}$$

(Liu-Wang)

$\hat{\Delta}_S$ is the δ-ring generated over

$$\mathbb{Z}_p[[x^{\frac{1}{p^\infty}}, y^{\frac{1}{p^\infty}}]]$$

by e modulo equation (1) and completed under the Nygaard filtration.

Generarstors of $\hat{\triangle}_S$

$$e - 1 = \frac{x^p - y^p}{p} + \frac{x^{2p} - x^p y^p}{p^2} + \dots$$

$$\delta(e - 1) = \frac{(x^p - y^p)^p - p^{p-1}(x^{p^2} - y^{p^2})}{p^{p+1}} + \frac{(x^p - y^p)^p x^p}{p^{p+1}} + \dots$$

$$\dots\dots$$

Hopf Algebroid Structure maps

unit

$$\eta_L(z) = x$$

$$\eta_R(z) = y$$

co-multiplication

$$\psi(x) = x \otimes 1$$

$$\psi(y) = 1 \otimes y$$

comudule structure on $TP_*(R)$ (Breuil-Kisin twist)

$$\eta_R(\sigma) = \epsilon^{-1}\sigma$$

where $\phi(\epsilon) = e\epsilon$.

Algebraic Tate Spectral Sequence

Using the Nygaard filtration on $TP_*(R)$, we have the spectral sequence:

$$E_1^{i,j,2k} = THH_i(R^{\otimes j}) \Rightarrow E_2^{i-2k,j}(TP(\mathbb{Z}_p)), i,j \geq 0, k \in \mathbb{Z}$$

and its mod p analog:

$$E_1^{i,j,2k} = (THH/p)_i(R^{\otimes j}) \Rightarrow E_2^{i-2k,j}((TP/p)(\mathbb{Z}_p)), i,j \geq 0, k \in \mathbb{Z}$$

It turns out that they are isomorphic to the Tate spectral sequences after the E_2-term.

The Modulo-p E_2-term

In the mod p algebraic Tate spectral sequence, we have

$$E_2^{*,*,*} = \mathbb{F}_p[f, g, \sigma^{\pm}]/(g^2)$$

such that

- $$f \in E_2^{2p,0,p}$$
  represented by z^p,
- $$g \in E_2^{2p,1,p}$$
  represented by $e - 1 = \frac{x^p - y^p}{p} + \ldots$,
- $$\sigma \in E_2^{0,0,-1}$$
  represented by σ.

The element $log(\frac{e}{\phi(e)^{\frac{1}{p}}})$

(Liu-Wang)

The element

$$log(\frac{e}{\phi(e)^{\frac{1}{p}}}) = log(e) - \frac{log(\phi(e))}{p}$$

lies in $\hat{\triangle}_S$.

$$log(\frac{e}{\phi(e)^{\frac{1}{p}}}) = \frac{x^p - y^p}{p} + \frac{x^{2p} - y^{2p}}{2p^2} + \cdots + \frac{(1 + p^{p-1})(x^{p^2} - y^{p^2})}{p^{p+1}} + \ldots$$

Tate d_p-differentials

$$x^p - y^p \doteq \frac{x^{2p} - y^{2p}}{p} + \dots \mod p$$

implies:

$$d_p(f) = fg$$

d_{p^2+p}-differentials

Applying Frobenius, we get

$$x^{p^2} - y^{p^2} \doteq \frac{x^{2p^2} - y^{2p^2}}{p} + \dots \mod p$$

Expanding

$$p^{2p} log(\frac{e}{\phi(e)^{\frac{1}{p}}})$$

we get

$$\frac{x^{2p^2} - y^{2p^2}}{p} \doteq \frac{x^{2p^2+p} - y^{2p^2+p}}{p} + \dots \mod p$$

These imply:

$$d_{p^2+p}(f^p) = f^{2p}g$$

Higher Differentials

Let

$$\psi(k) = \frac{p^k - 1}{p - 1}$$

By induction, we have:

$$d_{p\psi(k)}(f^{p^{k-1}}) \doteq g f^{p^{k-1} + \psi(k) - 1}$$

TC of integers

We can recover Bökstedt and Madsen's computations of $TC/p(\mathbb{Z}_p)$:

$TP/p(\mathbb{Z}_p)$ is generated by the classes:

$$(\sigma^{p-1}f)^i, (\sigma^p g)(\sigma^{p-1}f)^i, \sigma^{np^k}(\sigma^p g)(\sigma^{p-1}f)^j$$

with $i, k \geq 0$; $n \in \mathbb{Z}, p \nmid n$; $0 \leq j < \psi(k+1) - 1$.

$(TC/p)_*(\mathbb{Z}_p)$ is free over $\mathbb{F}_p[\mathfrak{v}_1]$ with generators

$$\{1, \gamma, a_0\gamma, a_m \text{ for } m = 0, 1, \ldots, p-1\}$$

lying in stem 0, -1, $2p-2$ and $2p-2m-1$ respectively.

Remark

The Bockstein sends $\mathfrak{v}_1$ to a_1, and $\mathfrak{v}_1 a_{p-1}$ to $a_0\gamma$.

Local fields (work in progress)

Let $f(x)$ be an Eisenstein polynomial, and $R = \mathbb{Z}[x]/f(x)$.
Consider the augmented cosimplicial ring spectrum

$$\mathbb{S} \to \mathbb{S}[z] \to \mathbb{S}[z_1, z_2] \to \dots$$

Applying $THH_{(-)}(R)$ we get

$$THH_{\mathbb{S}[z]}(R) \to THH_{\mathbb{S}[z_1,z_2]}(R) \to \dots$$

$$THH_{\mathbb{S}[z]}(R) \cong R[u]$$

$$THH_{\mathbb{S}[z_1,z_2]}(R) \cong R[u] \otimes \Gamma(\delta)$$

$$\eta_R(u) = u + f'(x)\delta$$

The Prismatic Hopf Algeboid

We set

$$A = \mathbb{Z}[x]$$

$$B = \mathbb{Z}[x_1, x_2, e, \delta(e), \dots]/(ef(x_1^p) - (x_1^p - x_2^p), \dots)$$

then (A, B) froms a Hopf algebraoid. If we can show $\pi_0 TP_{\mathbb{S}[z_1,z_2]}(R)$ is a δ-ring, then we have

- $\pi_0 TP_{\mathbb{S}[z_1,z_2]}(R) \cong B$.
- $\pi_* TP_{\mathbb{S}[z]}(R) \cong A[\sigma^{\pm}]$ is a B-comodule.
- We have the spectral sequence

$$Ext^j_B(A, \pi_i TP_{\mathbb{S}[z]}(R)) \Rightarrow TP_{i-j}(R)$$

- The above spectral sequence is consentrated in $j = 0, 1$ and collapses at the E_2-term.

Thanks!